Whole Numbers and Their Basic Properties

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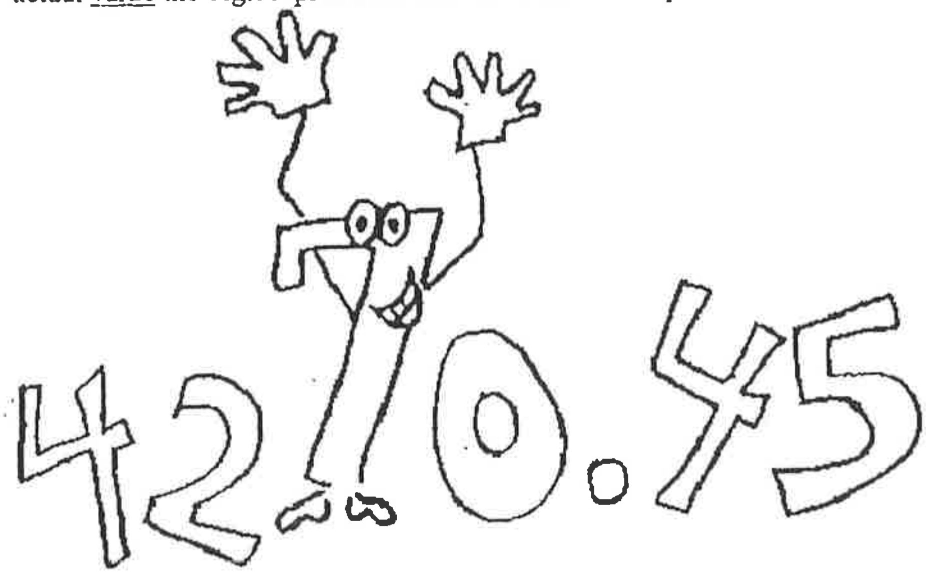
Whole Numbers

The whole numbers are the counting numbers and 0. The whole numbers are 0, 1, 2, 3, 4, 5, ...

Place Value

The position, or place, of a digit in a number written in standard form determines the actual value the digit represents. This table shows the place value for various positions:



Place (underlined)	Name of Position
1 000	Ones (units) position
1 000	Tens
1 000	Hundreds
1 000	Thousands
1 000 000	Ten thousands
1 000 000	Hundred Thousands
1 000 000	Millions
1 000 000 000	Ten Millions
1 000 000 000	Hundred millions
1 000 000 000	Billions

Example:

The number 721040 has a 7 in the hundred thousands place, a 2 in the ten thousands place, a one in the thousands place, a 4 in the tens place, and a 0 in both the hundreds and ones place.

Expanded Form

The expanded form of a number is the sum of its various place values.

Example:

9836 = 9000 + 800 + 30 + 6.

Ordering

Symbols are used to show how the size of one number compares to another. These symbols are < (less than), > (greater than), and = (equals.) For example, since 2 is smaller than 4 and 4 is larger than 2, we can write: 2 < 4, which says the same as 4 > 2 and of course, 4 = 4.

To compare two whole numbers, first put them in standard form. The one with more digits is greater than the other. If they have the same number of digits, compare the most significant digits (the leftmost digit of each number). The one having the larger significant digit is greater than the other. If the most significant digits are the same, compare the next pair of digits from the left. Repeat this until the pair of digits is different. The number with the larger digit is greater than the other.

Example: 402 has more digits than 42, so 402 > 42.

Example: 402 and 412 have the same number of digits. We compare the leftmost digit of each number: 4 in each case. Moving to the right, we compare the next two numbers: 0 and 1. Since 0 < 1, 402 < 412.

Rounding Whole Numbers

To round to the nearest ten means to find the closest number having all zeros to the right of the tens place. Note: when the digit 5, 6, 7, 8, or 9 appears in the ones place, round up; when the digit 0, 1, 2, 3, or 4 appears in the ones place, round down.

Examples:

Rounding 119 to the nearest ten gives 120.
Rounding 155 to the nearest ten gives 160.
Rounding 102 to the nearest ten gives 100.
Similarly, to round a number to any place value, we find the number with zeros in all of the places to the right of the place value being rounded to that is closest in value to the original number.

Examples:

Rounding 180 to the nearest hundred gives 200.
Rounding 150090 to the nearest hundred thousand gives 200000.

Rounding 1234 to the nearest thousand gives 1000.
Rounding is useful in making estimates of sums, differences, etc.

Example:

To estimate the sum 119360 + 500 to the nearest thousand, first round each number in the sum, resulting in a new sum of 119000 + 1000.. Then add to get the estimate of 120000.

Divisibility Tests

There are many quick ways of telling whether or not a whole number is divisible by certain basic whole numbers. These can be useful tricks, especially for large numbers.

Divisibility by 2
Divisibility by 3
Divisibility by 4
Divisibility by 5
Divisibility by 6
Divisibility by 8
Divisibility by 9
Divisibility by 10
Divisibility by 11
Divisibility by 12
Divisibility by 15
Divisibility by 16
Divisibility by 18
Divisibility by 20
Divisibility by 22
Divisibility by 25

Commutative Property of Addition and Multiplication

Addition and Multiplication are commutative: switching the order of two numbers being added or multiplied does not change the result.

Examples:

$100 + 8 = 8 + 100$
$100 \times 8 = 8 \times 100$

Associative Property

Addition and multiplication are associative: the order that numbers are grouped in addition and multiplication does not affect the result.

Examples:

$(2 + 10) + 6 = 2 + (10 + 6) = 18$
$2 \times (10 \times 6) = (2 \times 10) \times 6 = 120$

Distributive Property

The distributive property of multiplication over addition: multiplication may be distributed over addition.

Examples:

$10 \times (50 + 3) = (10 \times 50) + (10 \times 3)$
$3 \times (12+99) = (3 \times 12) + (3 \times 99)$

The Zero Property of Addition

Adding 0 to a number leaves it unchanged. We call 0 the additive identity.

Example:

$88 + 0 = 88$

The Zero Property of Multiplication

Multiplying any number by 0 gives 0.

Example:

$88 \times 0 = 0$
$0 \times 1003 = 0$

The Multiplicative Identity

We call 1 the multiplicative identity. Multiplying any number by 1 leaves the number unchanged.

Example:

$88 \times 1 = 88$

Order of Operations

The order of operations for complicated calculations is as follows:

1) Perform operations within parentheses.
2) Multiply and divide, whichever comes first, from left to right.
3) Add and subtract, whichever comes first, from left to right.

Example:

$1 + 20 \times (6 + 2) \div 2 =$
$1 + 20 \times 8 \div 2 =$
$1 + 160 \div 2 =$
$1 + 80 =$
$81.$

Divisibility by 2

A whole number is divisible by 2 if the digit in its units position is even, (either 0, 2, 4, 6, or 8).

Examples:

The number 84 is divisible by 2 since the digit in the units position is 4, which is even.
The number 333336 is divisible by 2 since the digit in the units position is 6, which is even.
The number 1297000 is divisible by 2 since the digit in the units position is 0, which is even.

Divisibility by 3

A whole number is divisible by 3 if the sum of all its digits is divisible by 3.

Examples:

The number 177 is divisible by three, since the sum of its digits is 15, which is divisible by 3.
The number 8882151 is divisible by three, since the sum of its digits is 33, which is divisible by 3.
The number 162345 is divisible by three, since the sum of its digits is 21, which is divisible by 3.

Decimals, Whole Numbers, and Exponents

Decimal numbers
Whole number portion
Expanded form of a decimal number
Adding decimals
Subtracting decimals
Comparing decimal numbers
Rounding decimal numbers
Estimating sums and differences
Multiplying decimal numbers
Dividing whole numbers, with remainders
Dividing whole numbers, with decimal portions
Dividing decimals by whole numbers
Dividing decimals by decimals
Exponents (powers of 2, 3, 4, ...)
Factorial notation
Square roots

Decimal Numbers

Decimal numbers such as 3.762 are used in situations which call for more precision than whole numbers provide.

As with whole numbers, a digit in a decimal number has a value which depends on the place of the digit. The places to the left of the decimal point are ones, tens, hundreds, and so on, just as with whole numbers. This table shows the decimal place value for various positions:

Note that adding extra zeros to the right of the last decimal digit does not change the value of the decimal number.

Place (underlined)	Name of Position
1.234567	Ones (units) position
1.234567	Tenths

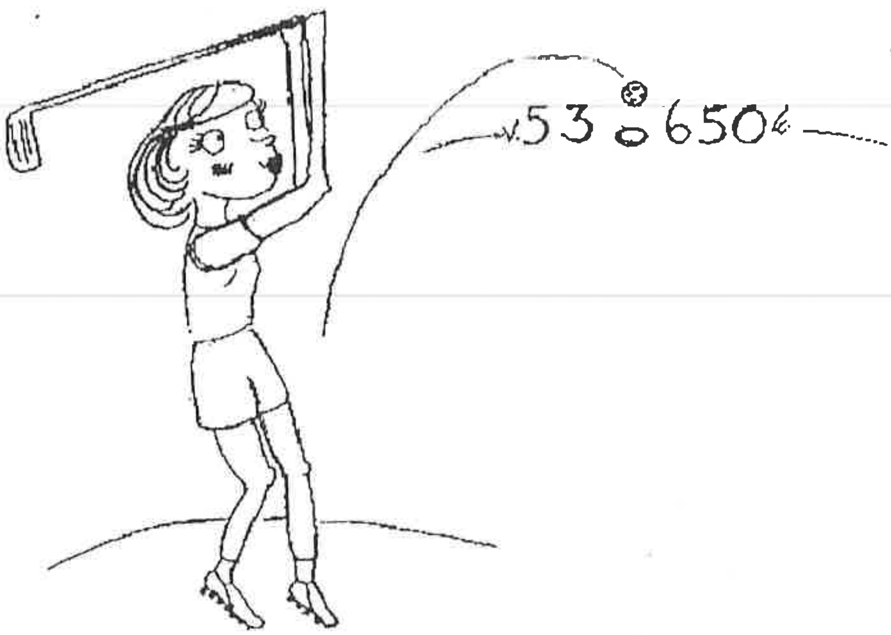
1.234567 Hundredths
1.234567 Thousandths
1.234567 Ten thousandths
1.234567 Hundred Thousandths
1.234567 Millionths

Example:

In the number 3.762, the 3 is in the ones place, the 7 is in the tenths place, the 6 is in the hundredths place, and the 2 is in the thousandths place.

Example:

The number 14.504 is equal to 14.50400, since adding extra zeros to the right of a decimal number does not change its



value.

Whole Number Portion

The whole number portion of a decimal number are those digits to the left of the decimal place.

Example:

In the number 23.65, the whole number portion is 23.

In the number 0.024, the whole number portion is 0.

Expanded Form of a Decimal Number

The expanded form of a decimal number is the number written as the sum of its whole number and decimal place values.

Example:

3 + 0.7 + 0.06 + 0.002 is the expanded form of the number 3.762.

100 + 3 + 0.06 is the expanded form of the number 103.06.

Adding Decimals

To add decimals, line up the decimal points and then follow the rules for adding or subtracting whole numbers, placing the decimal point in the same column as above.

When one number has more decimal places than another, use 0's to give them the same number of decimal places.

Example:

76.69 + 51.37

1) Line up the decimal points:

$$\begin{array}{r} 76.69 \\ +51.37 \end{array}$$

2) Then add.

$$\begin{array}{r} 76.69 \\ +\underline{51.37} \\ 128.06 \end{array}$$

Example:

12.924 + 3.6

1) Line up the decimal points:

12.924
+ 3.600

2) Then add.

12.924
+ 3.600
16.524

Subtracting Decimals

To subtract decimals, line up the decimal points and then follow the rules for adding or subtracting whole numbers, placing the decimal point in the same column as above.

When one number has more decimal places than another, use 0's to give them the same number of decimal places.

Example:

18.2 - 6.008

1) Line up the decimal points.

18.2
-
6.008

2) Add extra 0's, using the fact that 18.2 = 18.200

18.200
-
6.008

3) Subtract.

18.200
-
6.008
12.192

Comparing Decimal Numbers

Symbols are used to show how the size of one number compares to another. These symbols are < (less than), > (greater than), and = (equals). To compare the size of decimal numbers, we compare the whole number portions first. The larger decimal number is the one with the lager whole number portion. If the whole number parts are both equal, we compare the decimal portions of the numbers. The leftmost decimal digit is the most significant digit. Compare the pairs of digits in each decimal place, starting with the most significant digit until you find a pair that is different. The number with the larger digit is the larger number. Note that the number with the most digits is not necessarily the largest.

Example:

Compare 1 and 0.002. We begin by comparing the whole number parts: in this case 1>0, 0 being the whole number part of 0.002, and so 1>0.002.

Example:

Compare 0.402 and 0.412. The numbers 0.402 and 0.412 have the same number of digits, and their whole number parts are both 0. We compare the next most significant digit of each number, the digit in the tenths place, 4 in each case. Since they are equal, we go on to the hundredths place, and in this case, 0<1, so 0.402<0.412.

Example:

Compare 120.65 and 34.999. Comparing the whole number parts, 120>34, so 120.65>34.999.

Example:

Compare 12.345 and 12.097. Since the whole number parts are both equal, we compare the decimal portions starting with the tenths digit. Since 3>0, we have 12.345>12.097.

Note:

Remember that adding extra zeros to the right of a decimal does not change its value:

2.4 = 2.40 = 2.400 = 2.4000.

Rounding Decimal Numbers

To round a number to any decimal place value, we want to find the number with zeros in all of the lower places that is closest in value to the original number. As with whole numbers, we look at the digit to the right of the place we wish to round to. Note: When the digit 5, 6, 7, 8, or 9 appears in the ones place, round up; when the digit 0, 1, 2, 3, or 4 appears in the ones place, round down.

Examples:

Rounding 1.19 to the nearest tenth gives 1.2 (1.20).

Rounding 1.545 to the nearest hundredth gives 1.55.

Rounding 0.1024 to the nearest thousandth gives 0.102.

Rounding 1.80 to the nearest one gives 2.

Rounding 150.090 to the nearest hundred gives 200.

Rounding 4499 to the nearest thousand gives 4000.

Estimating Sums and Differences

We can use rounding to get quick estimates on sums and differences of decimal numbers. First round each number to the place value you choose, then add or subtract the rounded numbers to estimate the sum or difference.

Example:

To estimate the sum 119.36 + 0.56 to the nearest whole number, first round each number to the nearest one, giving us 119 + 1, then add to get 120.

Multiplying Decimal Numbers

Multiplying decimals is just like multiplying whole numbers. The only extra step is to decide how many digits to leave to the right of the decimal point. To do that, add the numbers of digits to the right of the decimal point in both factors.

Example:

4.032×4

We can multiply 4032 by 4 to get 16128. There are three decimal places in 4.032, so place the decimal three digits from the right:

$4.032 \times 4 = 16.128$

Example:

6.74×9.063

We can multiply 674 by 9063 to get 6108462. Then there are 5 decimal places: two in the number 6.74 and three in the number 9.063, so place the decimal five digits from the right:

$6.74 \times 9.063 = 61.08462.$

Dividing Whole Numbers, with Remainders

Example:

$1400 \div 7$..

Since $14 \div 7 = 2$, and 1400 is 100 times greater than 14, the answer is $2 \times 100 = 200$.

Many problems are similar to the above example, where the answer is easily obtained by adding on or taking off an appropriate number of 0's. Others are more complicated.

Example:

$4934 \div 6$. Use long division.

$$6\overline{)4934} \Rightarrow \begin{array}{r} 8 \\ 6\overline{)4934} \\ -4800 \\ 134 \end{array} \Rightarrow \begin{array}{r} 82 \\ 6\overline{)4934} \\ -4800 \\ 134 \\ -120 \\ 14 \end{array} \Rightarrow \begin{array}{r} 822 \\ 6\overline{)4934} \\ -4800 \\ 134 \\ -120 \\ 14 \\ -12 \\ 2 \end{array}$$

So the answer is 822 with a remainder of 2, written 822 R2.

To double-check that the answer is correct, multiply the quotient by the divisor and add the remainder:

$(822 \times 6) + 2 = 4932 + 2 = 4934.$

Dividing Whole Numbers, with Decimal Portions

Example:

Find 32 ÷ 6 to the nearest whole number.

32 ÷ 6 = 5 r2. 6 is the divisor; 2 is the remainder.

2 is closer to 0 than 6, so round down. The answer is 5.

Dividing Decimals by Whole Numbers

To divide a decimal by a whole number, use long division, and just remember to line up the decimal points:

Example:

13.44 ÷ 12.

```
     1.12
12)13.44
  -12
    14
   -12
     24
    -24
      0
```

When rounding an answer, divide one place further than the place you're rounding to, and round the result. Add 0's to the right of the number being divided, if necessary.

Example:

1.0 ÷ 6. Round to the nearest thousandth.

```
   0.16666
6 )1.00000
  -6
    40
   -36
     40
    -36
      40
     -36
       40
      -36
        ⋮
```

To round 0.16666 . . . to the nearest thousandth, we take 4 places to the right of the decimal point and round to 3 places. Here, we round 0.1666 to 0.167, the answer.

Dividing Decimals by Decimals

To divide by a decimal, multiply that decimal by a power of 10 great enough to obtain a whole number. Multiply the dividend by that same power of 10. Then the problem becomes one involving division by a whole number instead of division by a decimal.

$\sqrt{r^2} = \sqrt{n} = |r|$

We say the absolute value, because the notation $\sqrt{n}$ actually means the positive square root of n.

Example:

$\sqrt{(-3)^2} = \sqrt{(-3)\times(-3)} = \sqrt{9} = 3 = |-3|$

From the example above, we see that each positive number n actually has 2 numbers r that satisfy $r \times r = n$, one is positive, and the other is negative.

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Example:

$0.144 \div 0.12$

Multiplying the divisor (0.12) and the dividend (0.144) by 100, then dividing, gives the same result.

$0.12\overline{)0.144} = 12\overline{)14.4}$

The answer is 1.2.

Be aware that some problems are less difficult and do not require this procedure.

Example:

$6 \div 2.00$

This is the same as $6 \div 2$! The answer is 3.

Exponents (Powers of 2, 3, 4, ...)

Exponential notation is useful in situations where the same number is multiplied repeatedly.

The number being multiplied is called the base, and the exponent tells how many times the base is multiplied by itself.

Example:

$4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^6$

The base in this example is 4, the exponent is 6.

We refer to this as four to the sixth power, or four to the power of six.

Examples:

$2 \times 2 \times 2 = 2^3 = 8$

$1.1^2 = 1.1 \times 1.1 = 1.21$

$0.5^3 = 0.5 \times 0.5 \times 0.5 = 0.125$

$10^6 = 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 1000000$

Fractions

Prime numbers
Greatest common factor
Least common multiple
What is a fraction?
Equivalent fractions
Comparing fractions
Converting and reducing fractions
Lowest terms
Improper fractions
Mixed numbers
Converting mixed numbers to improper fractions
Converting improper fractions to mixed numbers
Writing a fraction as a decimal
Rounding a fraction to the nearest hundredth
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Adding and subtracting mixed numbers
Multiplying fractions and whole numbers
Multiplying fractions and fractions
Multiplying mixed numbers
Reciprocal
Dividing fractions
Dividing mixed numbers
Simplifying complex fractions
Repeating decimals

Prime Numbers

A whole number greater than one that is divisible by only 1 and itself. The numbers 2, 3, 5, 37, and 101 are some examples of prime numbers.

Greatest Common Factor

The greatest common factor of two or more whole numbers is the largest whole number that divides each of the numbers.

There are two methods of finding the greatest common factor of two numbers.

Method 1: List all the factors of each number, then list the common factors and choose the largest one.

Example:

36: 1, 2, 3, 4, 6, 9, 12, 18, 36

54: 1, 2, 3, 6, 9, 18, 27, 54

The common factors are: 1, 2, 3, 6, 9, and 18.

The greatest common factor is: 18.

Method 2: List the prime factors, then multiply the common prime factors.

Example:

$36 = \underline{2} \times 2 \times \underline{3} \times \underline{3}$

$54 = \underline{2} \times 3 \times \underline{3} \times \underline{3}$

The common prime factors are 2, 3, and 3.

The greatest common factor is $\underline{2} \times \underline{3} \times \underline{3} = 18$..

Least Common Multiple

The least common multiple of two or more nonzero whole numbers is the smallest whole number that is divisible by each of the numbers. There are two common methods for finding the least common multiple of 2 numbers.

Method 1:

List the multiples of each number, and look for the smallest number that appears in each list.

Example:

Find the least common multiple of 12 and 42. We list the multiples of each number:

12: 12, 24, 36, 48, 60, 72, 84, ...

42: 42, 84, 126, 168, 190, ...

We see that the number 84 is the smallest number that appears in each list.

Method 2:

Factor each of the numbers into primes. For each different prime number in either of the factorizations, follow these steps:

1. Count the number of times it appears in each of the factorizations.

2. Take the largest of these two counts.

3. Write down that prime number as many times as the count in step 2.

To find the least common multiple take the product of all of the prime numbers written down in steps 1, 2, and 3.

Example:

Find the least common multiple of 24 and 90. First, we find the prime factorization of each number.

$24 = 2 \times 2 \times 2 \times 3$

$90 = 2 \times 3 \times 3 \times 5$

The prime numbers 2, 3, and 5 appear in the factorizations. We follow steps 1 through 3 for each of these primes.

The number 2 occurs 3 times in the first factorization and 1 time in the second, so we will use three 2's.

The number 3 occurs 1 time in the first factorization and 2 times in the second, so we will use two 3's.

The number 5 occurs 0 times in the first factorization and 1 time in the second factorization, so we will use one 5.

The least common multiple is the product of three 2's, two 3's, and one 5.

$2 \times 2 \times 2 \times 3 \times 3 \times 5 = 360$

Example:

Find the least common multiple of 14 and 49. First, we find the prime factorization of each number.

$14 = 2 \times 7$

$49 = 7 \times 7$

The prime numbers 2 and 7 appear in the factorizations. We follow steps 1 through 3 for each of these primes.

The number 2 occurs 1 times in the first factorization and 0 times in the second, so we will use one 2.

The number 7 occurs 1 time in the first factorization and 2 times in the second, so we will use two 7's.

The least common multiple is the product of one 2 and two 7's.

$2 \times 7 \times 7 = 98$

Examples:

Some other least common multiples are listed below.

The least common multiple of 12 and 9 is 36.

The least common multiple of 6 and 18 is 18.

The least common multiple of 2, 3, 4, and 5 is 60.

What is a Fraction?

A fraction is a number that expresses part of a group.

Fractions are written in the form $\frac{a}{b}$ or *a/b*, where a and b are whole numbers, and the number b is not 0. For the purposes of these web pages, we will denote fractions using the notation *a/b*, though the preferred notation is generally $\frac{a}{b}$.

Converting and Reducing Fractions

For any fraction, multiplying the numerator and denominator by the same nonzero number gives an equivalent fraction. We can convert one fraction to an equivalent fraction by using this method.

Examples:

$1/2 = (1 \times 3)/(2 \times 3) = 3/6$

$2/3 = (2 \times 2)/(3 \times 2) = 4/6$

$3/5 = (3 \times 4)/(5 \times 4) = 12/20$

Another method of converting one fraction to an equivalent fraction is by dividing the numerator and denominator by a common factor of the numerator and denominator.

Examples:

$20/42 = (20 \div 2)/(42 \div 2) = 10/21$

$36/72 = (36 \div 3)/(72 \div 3) = 12/24$

$9/27 = (9 \div 3)/(27 \div 3) = 3/9$

When we divide the numerator and denominator of a fraction by their greatest common factor, the resulting fraction is an equivalent fraction in lowest terms.

Lowest Terms

A fraction is in lowest terms when the greatest common factor of its numerator and denominator is 1. There are two methods of reducing a fraction to lowest terms.

<u>Method 1:</u>

Divide the numerator and denominator by their greatest common factor.

$12/30 = (12 \div 6)/(30 \div 6) = 2/5$

<u>Method 2:</u>

Divide the numerator and denominator by any common factor. Keep dividing until there are no more common factors.

12/30 = (12 ÷ 2)/(30 ÷ 2) = 6/15 = (6 ÷ 3)/(15 ÷ 3) = 2/5

Improper Fractions

Improper fractions have numerators that are larger than or equal to their denominators.

Examples:

11/4, 5/5, and 13/2 are improper fractions.

Mixed Numbers

Mixed numbers have a whole number part and a fraction part.

Examples:

$2\frac{3}{4}$ and $6\frac{1}{2}$ are mixed numbers also written as 2 3/4 and 6 1/2. In these web pages, we denote mixed numbers in the form *a b/c*.

Converting Mixed Numbers to Improper Fractions

To change a mixed number into an improper fraction, multiply the whole number by the denominator and add it to the numerator of the fractional part.

Examples:

2 3/4 = ((2 × 4) + 3)/4 =11/4

6 1/2 = ((6 × 2) + 1)/2 = 13/2

Converting Improper Fractions to Mixed Numbers

To change an improper fraction into a mixed number, divide the numerator by the denominator. The remainder is the numerator of the fractional part.

Examples:

$11/4 = 11 \div 4 = 2\ r3 = 2\ 3/4$

$13/2 = 13 \div 2 = 6\ r1 = 6\ 1/2$

Writing a Fraction as a Decimal

Method 1 - Convert to an equivalent fraction whose denominator is a power of 10, such as 10, 100, 1000, 10000, and so on, then write in decimal form.

Examples:

$1/4 = (1 \times 25)/(4 \times 25) = 25/100 = 0.25$

$3/20 = (3 \times 5)/(20 \times 5) = 15/100 = 0.15$

$9/8 = (9 \times 125)/(8 \times 125) = 1125/1000 = 1.125$

Method 2 - Divide the numerator by the denominator. Round to the decimal place asked for, if necessary.

Example:

$13/4 = 13 \div 4 = 3.25$

Example:

Convert 3/7 to a decimal.

Round to the nearest thousandth.

We divide one decimal place past the place we need to round to, then round the result.

$3/7 = 3 \div 7 = 0.4285\ldots$

which equals 0.429 when rounded to the nearest thousandth.

Example:

Convert 4/9 to a decimal.

Round to the nearest hundredth.

We divide one decimal place past the place we need to round to, then round the result.

4/9 = 4 ÷ 9 = 0.4444...

which equals 0.44 when rounded to the nearest hundredth.

Rounding a Fraction to the Nearest Hundredth

Divide to the thousandths place. If the last digit is less than 5, drop it. This is particularly useful for converting a fraction to a percent, if we want to convert to the nearest percent.

1/3 = 1 ÷ 3 = 0.333... which rounds to 0.33

If the last digit is 5 or greater, drop it and round up.

2/7 = 2 ÷ 7 = 0.285 which rounds to 0.29

Adding and Subtracting Fractions

If the fractions have the same denominator, their sum is the sum of the numerators over the denominator. If the fractions have the same denominator, their difference is the difference of the numerators over the denominator. We do not add or subtract the denominators! Reduce if necessary.

Examples:

3/8 + 2/8 = 5/8

9/2 - 5/2 = 4/2 = 2

If the fractions have different denominators:
1) First, find the least common denominator.
2) Then write equivalent fractions using this denominator.
3) Add or subtract the fractions. Reduce if necessary.

Example:

3/4 + 1/6 = ?

The least common denominator is 12.

3/4 + 1/6 = 9/12 + 2/12 = 11/12.

Example:

9/10 - 1/2 = ?

The least common denominator is 10.

9/10 - 1/2 = 9/10 - 5/10 = 4/10 = 2/5.

Example:

2/3 + 2/7 = ?

The least common denominator is 21

2/3 + 2/7 = 14/21 + 6/21 = 20/21.

Adding and Subtracting Mixed Numbers

To add or subtract mixed numbers, simply convert the mixed numbers into improper fractions, then add or subtract them as fractions.

Example:

9 1/2 + 5 3/4 = ?

Converting each number to an improper fraction, we have 9 1/2 = 19/2 and 5 3/4 = 23/4.

We want to calculate 19/2 + 23/4. The LCM of 2 and 4 is 4, so

19/2 + 23/4 = 38/4 + 23/4 = (38 + 23)/4 = 61/4.

Converting back to a mixed number, we have 61/4 = 15 1/4.

The strategy of converting numbers into fractions when adding or subtracting is often useful, even in situations where one of the numbers is whole or a fraction.

Example:

13 - 1 1/3 = ?

In this situation, we may regard 13 as a mixed number without a fractional part. To convert it into a fraction, we look at the denominator of the fraction 4/3, which is 1 1/3 expressed as an improper fraction. The denominator is 3, and 13 = 39/3. So 13 - 1 1/3 = 39/3 - 4/3 = (39-4)/3 = 35/3, and 35/3 = 11 2/3.

Example:

5 1/8 - 2/3 = ?

This time, we may regard 2/3 as a mixed number with 0 as its whole part. Converting the first mixed number to an improper fraction, we have 5 1/8 = 41/8. The problem becomes

5 1/8 - 2/3 = 41/8 - 2/3 = 123/24 - 16/24 = (123 - 16)/24 = 107/24.

Converting back to a mixed number, we have 107/24 = 4 11/24.

Example:

92 + 4/5 = ?

This is easy. To express this as a mixed number, just put the whole number and the fraction side by side. The answer is 92 4/5.

Multiplying Fractions and Whole Numbers

To multiply a fraction by a whole number, write the whole number as an improper fraction with a denominator of 1, then multiply as fractions.

Example:

8 × 5/21 = ?

We can write the number 8 as 8/1. Now we multiply the fractions.

8 × 5/21 = 8/1 × 5/21 = (8 × 5)/(1 × 21) = 40/21

Example:

2/15 × 10 = ?

We can write the number 10 as 10/1. Now we multiply the fractions.

2/15 × 10 = 2/15 × 10/1 = (2 × 10)/(15 × 1) = 20/15 = 4/3

Multiplying Fractions and Fractions

When two fractions are multiplied, the result is a fraction with a numerator that is the product of the fractions' numerators and a denominator that is the product of the fractions' denominators.

Example:

$4/7 \times 5/11 = ?$

The numerator will be the product of the numerators: 4×5, and the denominator will be the product of the denominators: 7×11.

The answer is $(4 \times 5)/(7 \times 11) = 20/77$.

Remember that like numbers in the numerator and denominator cancel out.

Example:

$14/15 \times 15/17 = ?$

Since the 15's in the numerator and denominator cancel, the answer is

$14/15 \times 15/17 = 14/1 \times 1/17 = (14 \times 1)/(1 \times 17) = 14/17$

Example:

$4/11 \times 22/36 = ?$

In the solution below, first we cancel the common factor of 11 in the top and bottom of the product, then we cancel the common factor of 4 in the top and bottom of the product.

$4/11 \times 22/36 = 4/1 \times 2/36 = 1/1 \times 2/9 = 2/9$

Multiplying Mixed Numbers

To multiply mixed numbers, convert them to improper fractions and multiply.

Example:

$4\ 1/5 \times 2\ 2/3 = ?$.

Converting to improper fractions, we get $4\ 1/5 = 21/5$ and $2\ 2/3 = 8/3$. So the answer is

Repeating Decimals

Every fraction can be written as a decimal.

For example, 1/3 is 1 divided by 3.

If you use a calculator to find 1 ÷ 3, the calculator returns 0.333333... This is called a repeating decimal. To represent the idea that the 3's repeat forever, one uses a horizontal bar (overstrike) as shown below:

$0.\overline{3} = 0.333333...$

$1.412412412412... = 1.\overline{412}$

$104.6278787878... = 104.62\overline{78}.$

Example:

What is the repeating decimal for 1/7 ? Dividing 7 into 1, we get 0.142857142..., and we see the pattern begin to repeat with the second 1, so $\frac{1}{7} = 0.\overline{142857}$

Visit the Math League

Percent and Probability

Percent
What is a percent?
Percent as a fraction
Percent as a decimal
Estimating percents
Interest
Simple interest
Compound interest
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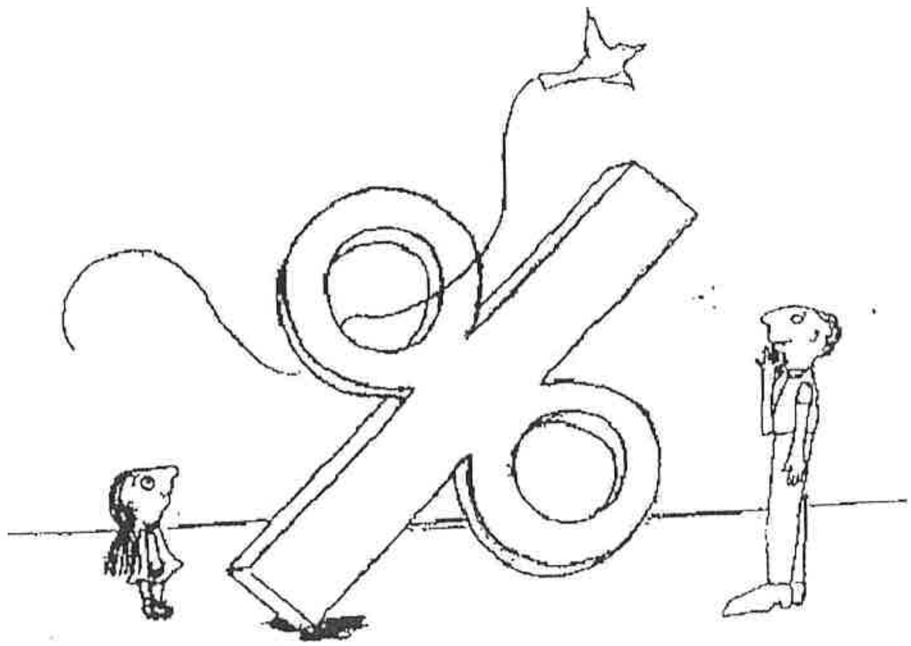
Percent discount

What is a Percent?

A percent is a ratio of a number to 100. A percent can be expressed using the percent symbol %.

Example: 10 percent or 10% are both the same, and stand for the ratio 10:100.



Percent as a fraction

A percent is equivalent to a fraction with denominator 100.

Example: 5% of something = 5/100 of that thing.

Example: 2 1/2% is equal to what fraction?
Answer:
2 1/2% = (2 1/2)/100 = 5/200 = 1/40

Example: 52% most nearly equals which one of 1/2, 1/4, 2, 8, or 1/5?
Answer: 52% = 52/100. This is very close to 50/100, or 1/2.

Example: 13/25 is what %?
We want to convert 13/25 to a fraction with 100 in the denominator:
13/25 = (13 × 4)/(25 × 4) = 52/100, so 13/25 = 52%.
Alternatively, we could say: Let 13/25 be n%, and let us find n. Then 13/25 = n/100, so cross multiplying, 13 × 100 = 25 × n, so 25n = 13 × 100 = 1300. Then 25n ÷ 25 = 1300 ÷ 25, so n = 1300 ÷ 25 = 52. So 13/25 = n% = 52%.

Example: 8/200 is what %?
Method 1: 8/200 = (4 × 2)/(100 × 2), so 8/200 = 4/100 = 4%.
Method 2: Let 8/200 be n%. Then 8/200 = n/100, so 200 × n = 800, and 200n ÷ 200 = 800 ÷ 200 = 4, so n% = 4%.

Example: Write 80% as a fraction in lowest terms.
80% = 80/100, which is equal to 4/5 in lowest terms.

Percent as a decimal

Percent and hundredths are basically equivalent. This makes conversion between percent and decimals very easy.

To convert from a decimal to a percent, just move the decimal 2 places to the right. For example, 0.15 = 15 hundredths = 15%.

Example:

0.0006 = 0.06%

Converting from percent to decimal form is similar, only you move the decimal point 2 places to the left. You must also be sure, before doing this, that the percentage itself is expressed in decimal form, without fractions.

Example:

Express 3% in decimal form. Moving the decimal 2 to the left (and adding in 0's to the left of the 3 as place holders,) we get 0.03.

Example:

Express 97 1/4% in decimal form. First we write 97 1/4 in decimal form: 97.25. Then we move the decimal 2 places to the left to get 0.9725, so 97 1/4% = 0.9725. This makes sense, since 97 1/4% is nearly 100%, and 0.9725 is nearly 1.

Estimating percents

When estimating percents, it is helpful to remember the fractional equivalent of some simple percents.

100% = 1
(100% of any number equals that number.)

50% = 1/2 = 0.5
(50% of any number equals half of that number.)

25% = 1/4 = 0.25
(25% of any number equals one-fourth of that number.)

10% = 1/10 = 0.1
(10% of any number equals one-tenth of that number.)

1% = 1/100 = 0.01
(1% of any number equals one-hundredth of that number.)

Because it is very easy to switch between a decimal and a percent, estimating a percent is as easy as estimating a fraction as a decimal, and converting to a percent by multiplying by 100.

Example:

Estimate 19 as a percent of 80.
As a fraction, 19/80 ≅ 20/80 = 1/4 =0.25 = 25%. The step used to estimate the percent occurred when we estimated 19/80 as 20/80.
The exact percent is actually 23.75%, so the estimate of 25% is only 1.25% off. (About 1 part in 100.)

Example:

Estimate 7 as a percent of 960.
As a fraction, 7/960 ≅ 7/100 = 0.007 = 0.7%. The step used to estimate the percent occurred when we estimated 7/960 as 7/1000.
The exact percent, to the nearest thousandth of a percent, is actually 0.729%.
To estimate the percent of a number, we may convert the percent to a fraction, if useful, to estimate the percent.

Example:

Estimate 13% of 72.
Twice 13% is 26%, which is very close to 25%, and 25%=1/4. We may multiply both sides by 1/2 to get an estimate for 13%: 13% ≅ 12.5% = 1/2 × 25% = 1/2 × 1/4 = 1/8.
Using our estimate of 1/8 for 13%, 1/8 × 72 = 9, so we get an estimate of 9 for 13% of 72.
If we had calculated this exactly, 13% of 72 equals 9.36. It may look like we did a lot more work to get the estimate of 9 that just multiplying 72 by 0.13, but with practice, keeping in mind some simple percents and the fractions they are equal to will enable you to estimate some number combinations very quickly.

Example:

Estimate 9.6% of 51.
Method 1: We could estimate 9.6% of 50. It would be easy to estimate 9.6% of 100,

which is just 9.6. Since 50 is half of 100, we can just take half of 9.6, which is 4.8. The actual value of 9.6% of 51 is 4.896, so an estimate of 4.8 is pretty good.

Method 2: We could estimate 10% of 51, which is just 5.1. This is not as close an estimate as method 1, but is still a good estimate of the actual answer of 4.896.

Interest

Interest is a fee paid to borrow money. It is usually charged as a percent of the total amount borrowed. The percent charged is called the interest rate. The amount of money borrowed is called the principal. There are two types of interest, simple interest and compound interest.

Example: A bank charges 7% interest on a $1000 loan. It will cost the borrower 7% of $1000, which is $70, for each year the money is borrowed. Note that when the loan is up, the borrower must pay back the original $1000.

Simple Interest

Simple interest is interest figured on the principal only, for the duration of the loan. Figure the interest on the loan for one year, and multiply this amount by the number of years the money is borrowed for.

Example: A bank charges 8% simple interest on a $600 loan, which is to be paid back in two years. It will cost the borrower 8% of $600, which is $48, for each year the money is borrowed. Since it is borrowed for two years, the total charge for borrowing the money will be $96. After the two years the borrower will still have to pay back the original $600.

Compound Interest

Compound interest is interest figured on the principal and any interest owed from previous years. The interest charged the first year is just the interest rate times the amount of the loan. The interest charged the second year is the interest rate, times the sum of the loan and the interest from the first year. The interest charged the third year is the interest rate, times the sum of the loan and the first two years' interest amounts. Continue figuring the interest in this way for any additional years of the loan.

Example: A bank charges 8% compound interest on a $600 loan, which is to be paid back in two years. It will cost the borrower 8% of $600 the first year, which is $48. The second year, it will cost 8% of $600 + $48 = $648, which is $51.84. The total amount of interest owed after the two years is $48 + $51.84 = $99.84. Note that this is more than the $96 that would be owed if the bank was charging simple interest.

Example: A bank charges 4% compound interest on a $1000 loan, which is to be paid back in three years. It will cost the borrower 4% of $1000 the first year, which is $40. The second year, it will cost 4% of $1000 + $40 = $1040, which is $41.60. The third year, it will cost 4% of $1040 + $41.60 = $1081.60, which is $43.26 (with rounding). The total amount of interest owed after the three years is $40 + $41.60 + 43.26 = $124.86.

Percent increase and decrease

Percent increase and decrease of a value measure how that value changes, as a percentage of its original value.

Example: A collectors' comic book is worth $120 in 1994, and in 1995 its value is $132. The change is $132 - $120 = $12, an increase in price of $12; since $12 is 10% of $120, we say its value increased by 10% from 1994 to 1995.

Example: A bakery makes a chocolate cake that has 8 grams of fat per slice. A new change in the recipe lowers the fat to 6 grams of fat per slice. The change is 8g - 6g = 2g, a decrease of 2 grams; since 2 grams is 25% of 8, we say that the new cake recipe has 25% less fat, or a 25% decrease in fat.

Example: Amy is training for the 1500 meter run. When she started training she could run 1500 meters in 5 minutes and 50 seconds. After a year of practice her time decreased by 8%. How fast can she run the race now? Her old time was 5 × 60 + 50 = 350 seconds, and 8% of 350 is 28, so she can run the race in 350 - 28 = 322 seconds (5 minutes and 22 seconds).

Example: A fishing magazine sells 110000 copies each month. The company's president wants to increase the sales by 6%. How many extra magazines would they have to sell to reach this goal? This problem is easy, since it only asks for the change in sales: 6% of 110000 equals 6600 more magazines.

Percent Discount

A discount is a decrease in price, so percent discount is the percent decrease in price.

Example: Chocolate bars normally cost 80 cents each, but are on sale for 40 cents each, which is 50% of 80, so the chocolate is on sale at a 50% discount.

Example: A compact disc that sells for \$12 is on sale at a 20% discount. How much does the disc cost on sale? The amount of the discount is 20% of \$12, which is \$2.40, so the sale price is \$12.00 - \$2.40 = \$9.60.

Example: Movie tickets sell for \$8.00 each, but if you buy 4 or more you get \$1.00 off each ticket. What percent discount is this? We figure \$1 as a percentage of \$8: \$1.00/\$8.00 × 100% = 12.5%, so this is a 12.5% discount.

Integers

Positive and negative integers
The number line
Absolute value of an integer
Adding integers
Subtracting integers
Multiplying integers
Dividing integers
Integer coordinates
Comparing integers

Positive and Negative Integers

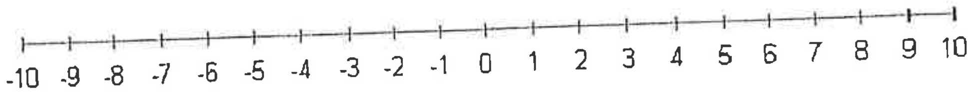
Positive integers are all the whole numbers greater than zero: 1, 2, 3, 4, 5, ... Negative integers are all the opposites of these whole numbers: -1, -2, -3, -4, -5, … . We do not consider zero to be a positive or negative number. For each positive integer, there is a negative integer, and these integers are called opposites. For example, -3 is the opposite of 3, -21 is the opposite of 21, and 8 is the opposite of -8. If an integer is greater than zero, we say that its *sign* is positive. If an integer is less than zero, we say that its *sign* is negative.

Example:

Integers are useful in comparing a direction associated with certain events. Suppose I take five steps forwards: this could be viewed as a positive 5. If instead, I take 8 steps *backwards*, we might consider this a -8. Temperature is another way negative numbers are used. On a cold day, the temperature might be 10 degrees below zero Celsius, or -10°*C*.

The Number Line

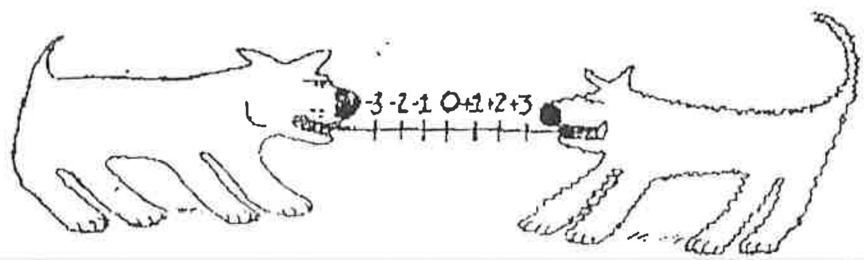
The number line is a line labeled with the integers in increasing order from left to right, that extends in both directions:



For any two different places on the number line, the integer on the right is greater than the integer on the left.

Examples:

9 > 4, 6 > -9, -2 > -8, and 0 > -5



Absolute Value of an Integer

The number of units a number is from zero on the number line. The absolute value of a number is always a positive number (or zero). We specify the absolute value of a number *n* by writing *n* in between two vertical bars: $|n|$.

Examples:

$|6| = 6$
$|-12| = 12$
$|0| = 0$
$|1234| = 1234$
$|-1234| = 1234$

Adding Integers

1) When adding integers of the same sign, we add their absolute values, and give the result the same sign.

Examples:

2 + 5 = 7
(-7) + (-2) = -(7 + 2) = -9
(-80) + (-34) = -(80 + 34) = -114

2) When adding integers of the opposite signs, we take their absolute values, subtract the smaller from the larger, and give the result the sign of the integer with the larger absolute value.

Example:

8 + (-3) = ?
The absolute values of 8 and -3 are 8 and 3. Subtracting the smaller from the larger gives 8 - 3 = 5, and since the larger absolute value was 8, we give the result the same sign as 8, so 8 + (-3) = 5.

Example:

8 + (-17) = ?
The absolute values of 8 and -17 are 8 and 17.
Subtracting the smaller from the larger gives 17 - 8 = 9, and since the larger absolute value was 17, we give the result the same sign as -17, so 8 + (-17) = -9.

Example:

-22 + 11 = ?
The absolute values of -22 and 11 are 22 and 11. Subtracting the smaller from the larger gives 22 - 11 = 11, and since the larger absolute value was 22, we give the result the same sign as -22, so -22 + 11 = -11.

Example:

53 + (-53) = ?

The absolute values of 53 and -53 are 53 and 53. Subtracting the smaller from the larger gives 53 - 53 =0. The sign in this case does not matter, since 0 and -0 are the same. Note that 53 and -53 are opposite integers. All opposite integers have this property that their sum is equal to zero. Two integers that add up to zero are also called additive inverses.

Subtracting Integers

Subtracting an integer is the same as adding its opposite.

Examples:

In the following examples, we convert the subtracted integer to its opposite, and add the two integers.
7 - 4 = 7 + (-4) = 3
12 - (-5) = 12 + (5) = 17
-8 - 7 = -8 + (-7) = -15
-22 - (-40) = -22 + (40) = 18

Note that the result of subtracting two integers could be positive or negative.

Multiplying Integers

To multiply a pair of integers if both numbers have the same sign, their product is the product of their absolute values (their product is positive). If the numbers have opposite signs, their product is the *opposite* of the product of their absolute values (their product is negative). If one or both of the integers is 0, the product is 0.

Examples:

In the product below, both numbers are positive, so we just take their product.
$4 \times 3 = 12$

In the product below, both numbers are negative, so we take the product of their absolute values.
$(-4) \times (-5) = |-4| \times |-5| = 4 \times 5 = 20$

In the product of $(-7) \times 6$, the first number is negative and the second is positive, so we take the product of their absolute values, which is $|-7| \times |6| = 7 \times 6 = 42$, and give this result a negative sign: -42, so $(-7) \times 6 = -42$.

In the product of $12 \times (-2)$, the first number is positive and the second is negative, so we take the product of their absolute values, which is $|12| \times |-2| = 12 \times 2 = 24$, and give this result a negative sign: -24, so $12 \times (-2) = -24$.

To multiply any number of integers:

1. Count the number of negative numbers in the product.
2. Take the product of their absolute values.
3. If the number of negative integers counted in step 1 is even, the product is just the product from step 2, if the number of negative integers is odd, the product is the opposite of the product in step 2 (give the product in step 2 a negative sign). If any of the integers in the product is 0, the product is 0.

Example:

4 × (-2) × 3 × (-11) × (-5) = ?

Counting the number of negative integers in the product, we see that there are 3 negative integers: -2, -11, and -5. Next, we take the product of the absolute values of each number:
4 × |-2| × 3 × |-11| × |-5| = 1320.
Since there were an odd number of integers, the product is the opposite of 1320, which is -1320, so
4 × (-2) × 3 × (-11) × (-5) = -1320.

Dividing Integers

To divide a pair of integers if both integers have the same sign, divide the absolute value of the first integer by the absolute value of the second integer.
To divide a pair of integers if both integers have different signs, divide the absolute value of the first integer by the absolute value of the second integer, and give this result a negative sign.

Examples:

In the division below, both numbers are positive, so we just divide as usual.
4 ÷ 2 = 2.

In the division below, both numbers are negative, so we divide the absolute value of the first by the absolute value of the second.
(-24) ÷ (-3) = |-24| ÷ |-3| = 24 ÷ 3 = 8.

In the division (-100) ÷ 25, both number have different signs, so we divide the absolute value of the first number by the absolute value of the second, which is |-100| ÷ |25| = 100 ÷ 25 = 4, and give this result a negative sign: -4, so (-100) ÷ 25 = -4.

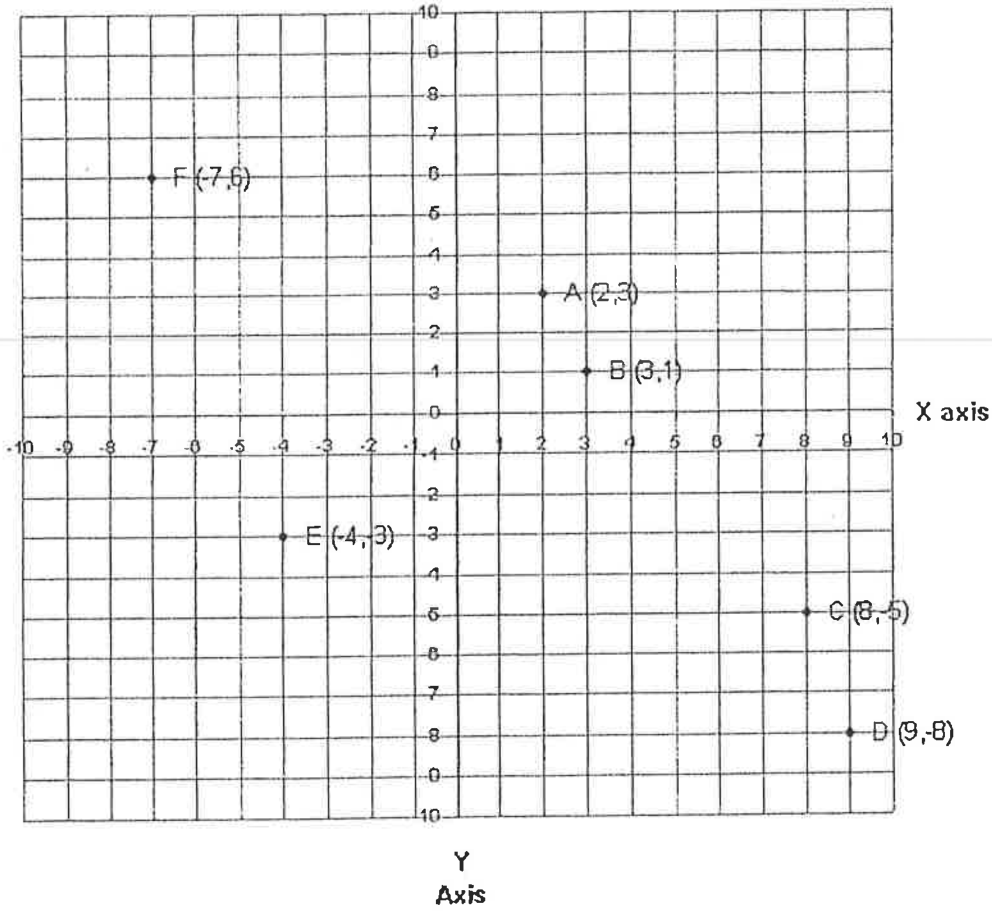
In the division 98 ÷ (-7), both number have different signs, so we divide the absolute value of the first number by the absolute value of the second, which is |98| ÷ |-7| = 98 ÷ 7 = 14, and give this result a negative sign: -14, so 98 ÷ (-7) = -14.

Integer coordinates

Integer coordinates are pairs of integers that are used to determine points in a grid, relative to a special point called the origin. The origin has coordinates (0,0). We can think of the origin as the center of the grid or the starting point for finding all other points. Any other point in the grid has a pair of coordinates (x,y). The x value or x-coordinate tells how many steps left or right the point is from the point (0,0), just like on the number line (negative is left of the origin, positive is right of the origin). The y value or y-coordinate tells how many steps up or down the point is from the point (0,0), (negative is down from the origin, positive is up from the origin). Using coordinates, we may give the location of any point in the grid we like by simply using a pair of numbers.

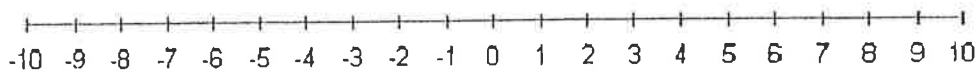
Example:

The origin below is where the x-axis and the y-axis meet. Point A has coordinates (2,3), since it is 2 units to the right and 3 units up from the origin. Point B has coordinates (3,1), since it is 3 units to the right, and 1 unit up from the origin. Point C has coordinates (8,-5), since it is 8 units to the right, and 5 units down from the origin. Point D has coordinates (9,-8); it is 9 units to the right, and 8 units down from the origin. Point E has coordinates (-4,-3); it is 4 units to the left, and 3 units down from the origin. Point F has coordinates (-7,6); it is 7 units to the left, and 6 units up from the origin.



Comparing Integers

We can compare two different integers by looking at their positions on the number line. For any two different places on the number line, the integer on the right is greater than the integer on the left. Note that every positive integer is greater than any negative integer.



Examples:

9 > 4, 6 > -9, -2 > -8, and 0 > -5
-2 < 1, 8 < 11, -7 < -5, and -10 < 0

Visit the Math League

Positive and negative numbers

About positive and negative numbers
The number line
Absolute value of positive and negative numbers
Adding positive and negative numbers
Subtracting positive and negative numbers
Multiplying positive and negative numbers
Dividing positive and negative numbers
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Reciprocals of negative numbers

About Positive and Negative Numbers

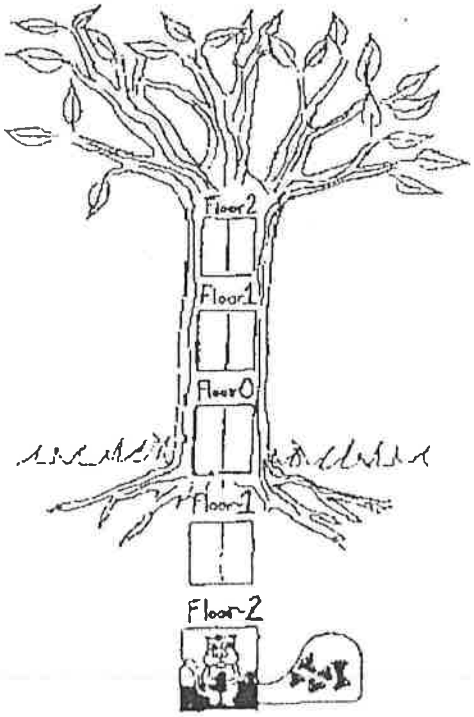
Positive numbers are any numbers greater than zero, for example: 1, 2.9, 3.14159, 40000, and 0.0005. For each positive number, there is a negative number that is its opposite. We write the opposite of a positive number with a negative or minus sign in front of the number, and call these numbers negative numbers. The opposites of the numbers in the list above would be: -1, -2.9, -3.14159, -40000, and -0.0005. Negative numbers are less than zero (see the number line for a more complete explanation of this). Similarly, the opposite of any negative number is a positive number. For example, the opposite of -12.3 is 12.3.
We do not consider zero to be a positive or negative number.
The sum of any number and its opposite is 0.
The *sign* of a number refers to whether the number is positive or negative, for example, the sign of -3.2 is negative, and the sign of 442 is positive.
We may also write positive and negative numbers as fractions or mixed numbers.

The following fractions are all equal:

(-1)/3, 1/(-3), -(1/3) and - 1/3.

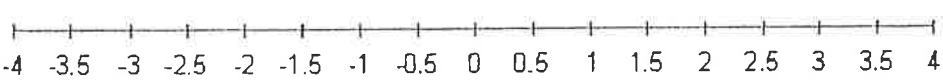
The following mixed numbers are all equal:

-1 1/6, -(1 1/6), (-7)/6, 7/(-6), and - 7/6.



The Number Line

The number line is a line labeled with positive and negative numbers in increasing order from left to right, that extends in both directions. The number line shown below is just a small piece of the number line from -4 to 4.



For any two different places on the number line, the number on the right is greater than the number on the left.

Examples:

$4 > -2$, $1 > -0.5$, $-2 > -4$, and $0 > -15$

Absolute Value of Positive and Negative Numbers

The number of units a number is from zero on the number line. The absolute value of a number is always a positive number (or zero). We specify the absolute value of a number *n* by writing *n* in between two vertical bars: $|n|$.

Examples:

$|6| = 6$
$|-0.004| = 0.004$
$|0| = 0$
$|3.44| = 3.44$
$|-3.44| = 3.44$
$|-10000.9| = 10000.9$

Adding Positive and Negative Numbers

1) When adding numbers of the same sign, we add their absolute values, and give the result the same sign.

Examples:

2 + 5.7 = 7.7
(-7.3) + (-2.1) = -(7.3 + 2.1) = -9.4
(-100) + (-0.05) = -(100 + 0.05) = -100.05

2) When adding numbers of the opposite signs, we take their absolute values, subtract the smaller from the larger, and give the result the sign of the number with the larger absolute value.

Example:

7 + (-3.4) = ?
The absolute values of 7 and -3.4 are 7 and 3.4. Subtracting the smaller from the larger gives 7 - 3.4 = 3.6, and since the larger absolute value was 7, we give the result the same sign as 7, so 7 + (-3.4) = 3.6.

Example:

8.5 + (-17) = ?
The absolute values of 8.5 and -17 are 8.5 and 17. Subtracting the smaller from the larger gives 17 - 8.5 = 8.5, and since the larger absolute value was 17, we give the result the same sign as -17, so 8.5 + (-17) = -8.5.

Example:

-2.2 + 1.1 = ?
The absolute values of -2.2 and 1.1 are 2.2 and 1.1. Subtracting the smaller from the larger gives 2.2 - 1.1 = 1.1, and since the larger absolute value was 2.2, we give the result the same sign as -2.2, so -2.2 + 1.1 = -1.1.

Example:

6.93 + (-6.93) = ?
The absolute values of 6.93 and -6.93 are 6.93 and 6.93. Subtracting the smaller from the larger gives 6.93 - 6.93 = 0. The sign in this case does not matter, since 0 and -0 are the same. Note that 6.93 and -6.93 are opposite numbers. All opposite numbers have this property that their sum is equal to zero. Two numbers that add up to zero are also called additive inverses.

Subtracting Positive and Negative Numbers

Subtracting a number is the same as adding its opposite.

Examples:

In the following examples, we convert the subtracted number to its opposite, and add the two numbers.
7 - 4.4 = 7 + (-4.4) = 2.6
22.7 - (-5) = 22.7 + (5) = 27.7
-8.9 - 1.7 = -8.9 + (-1.7) = -10.6
-6 - (-100.6) = -6 + (100.6) = 94.6

Note that the result of subtracting two numbers can be positive or negative, or 0.

Multiplying Positive and Negative Numbers

To multiply a pair of numbers if both numbers have the same sign, their product is the product of their absolute values (their product is positive). If the numbers have opposite signs, their product is the *opposite* of the product of their absolute values (their product is negative). If one or both of the numbers is 0, the product is 0.

Examples:

In the product below, both numbers are positive, so we just take their product.
0.5 × 3 = 1.5

In the product below, both numbers are negative, so we take the product of their absolute values.
(-1.1) × (-5) = |-1.1| × |-5| = 1.1 × 5 = 5.5

In the product of (-3) × 0.7, the first number is negative and the second is positive, so we take the product of their absolute values, which is |-3| × |0.7| = 3 × 0.7 = 2.1, and give this result a negative sign: -2.1, so (-3) × 0.7 = -2.1

In the product of 21 × (-3.1), the first number is positive and the second is negative, so we take the product of their absolute values, which is |21| × |-3.1| = 21 × 3.1 = 65.1, and give this result a negative sign: -65.1, so 21 × (-3.1) = -65.1.

To multiply any number of numbers:

1. Count the number of negative numbers in the product.
2. Take the product of their absolute values.
3. If the number of negative numbers counted in step 1 is even, the product is just the product from step 2, if the number of negative numbers is odd, the product is the opposite of the product in step 2 (give the product in step 2 a negative sign). If any of the numbers in the product is 0, the product is 0.

Example:

2 × (-1.1) × 5 (-1.2) × (-9) = ?
Counting the number of negative numbers in the product, we see that there are 3 negative numbers: -1.1, -1.2, and -9. Next, we take the product of the absolute values of each number: 2 × |-1.1| × 5 × |-1.2| × |-9| = 2 × 1.1 × 5 × 1.2 × 9 = 118.8
Since there were an odd number of numbers, the product is the opposite of 118.8, which is -118.8, so 2 × (-1.1) × 5 (-1.2) × (-9) = -118.8.

Dividing Positive and Negative Numbers

To divide a pair of numbers if both numbers have the same sign, divide the absolute value of the first number by the absolute value of the second number.
To divide a pair of numbers if both numbers have different signs, divide the absolute value of the first number by the absolute value of the second number, and give this result a negative sign.

Examples:

In the division below, both numbers are positive, so we just divide as usual.
7 ÷ 2 = 3.5

In the division below, both numbers are negative, so we divide the absolute value of the first by the absolute value of the second.
(-2.4) ÷ (-3) = |-2.4| ÷ |-3| = 2.4 ÷ 3 = 0.8

In the division (-1) ÷ 2.5, both number have different signs, so we divide the absolute value of the first number by the absolute value of the second, which is |-1| ÷ |2.5| = 1 ÷ 2.5 = 0.4, and give this result a negative sign: -0.4, so (-1) ÷ 2.5 = -0.4.

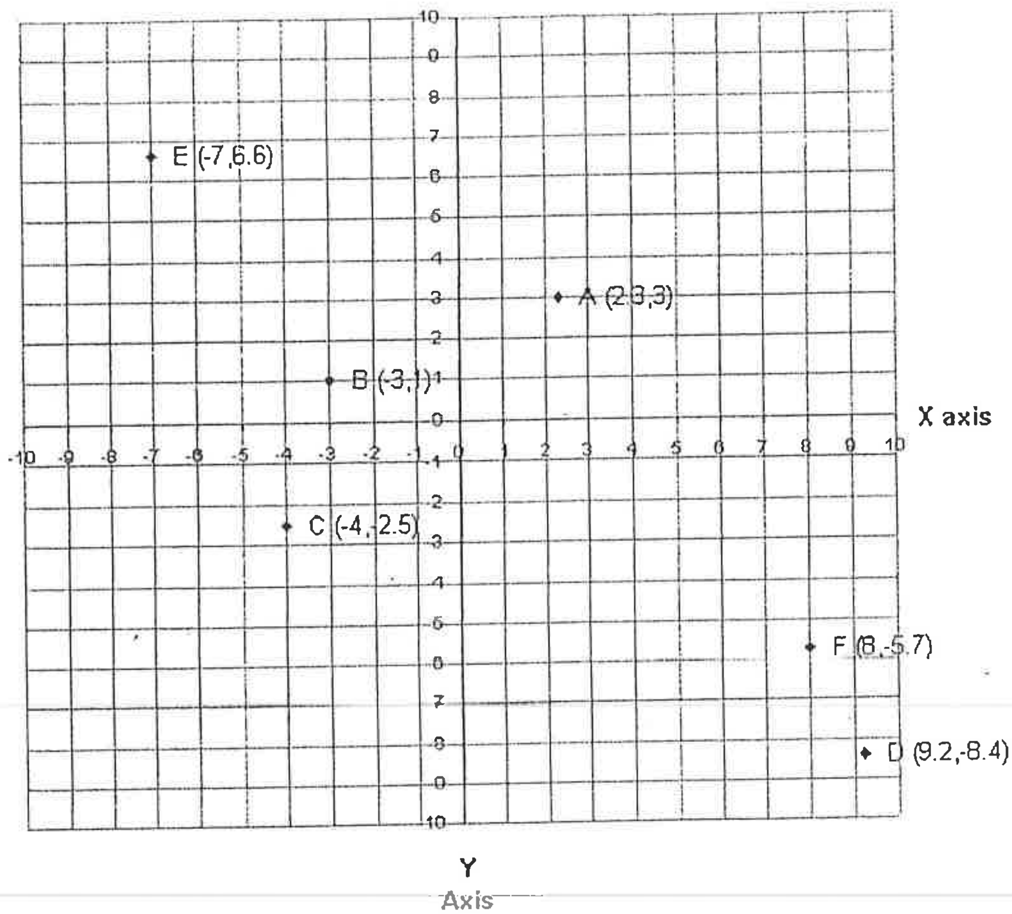
In the division 9.8 ÷ (-0.7), both number have different signs, so we divide the absolute value of the first number by the absolute value of the second, which is |9.8| ÷ |-0.7| = 9.8 ÷ 0.7 = 14, and give this result a negative sign: -14, so 9.8 ÷ (-0.7) = -14.

Coordinates

Number coordinates are pairs of numbers that are used to determine points in a grid, relative to a special point called the origin. The origin has coordinates (0,0). We can think of the origin as the center of the grid or the starting point for finding all other points. Any other point in the grid has a pair of coordinates (x,y). The x value or x-coordinate tells how many steps left or right the point is from the point (0,0), just like on the number line (negative is left of the origin, positive is right of the origin). The y value or y-coordinate tells how many steps up or down the point is from the point (0,0), (negative is down from the origin, positive is up from the origin). Using coordinates, we may give the location of any point in the grid we like by simply using a pair of numbers.

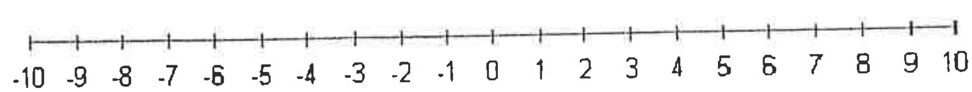
Example:

The origin below is where the x-axis and the y-axis meet. Point A has coordinates (2.3,3), since it is 2.3 units to the right and 3 units up from the origin. Point B has coordinates (-3,1), since it is 3 units to the left, and 1 unit up from the origin. Point C has coordinates (-4,-2.5), since it is 4 units to the left, and 2.5 units down from the origin. Point D has coordinates (9.2,-8.4); it is 9 units to the right, and 8.4 units down from the origin. Point E has coordinates (-7,6.6); it is 7 units to the left, and 6.6 units up from the origin. Point F has coordinates (8,-5.7); it is 8 units to the right, and 5.7 units down from the origin.



Comparing Positive and Negative Numbers

We can compare two different numbers by looking at their positions on the number line. For any two different places on the number line, the number on the right is greater than the number on the left. Note that every positive number is greater than any negative number.



Examples:

9.1 > 4, 6 > -9.3, -2 > -8, and 0 > -5.5
-2 < -13, -1 < -0.5, -7 < -5, and -10 < 0.1

Reciprocals of Negative Numbers

The reciprocal of a positive or negative fraction is obtained by switching its numerator and denominator, the sign of the new fraction remains the same. To find the reciprocal of a mixed number, first convert the mixed number to an improper fraction, then switch the numerator and denominator of the improper fraction. Notice that when you multiply negative fractions with their reciprocals, the product is always 1 (**NOT -1**).

Examples:

What is the reciprocal of -2/7? We just switch the numerator and denominator, and keep the same sign: -7/2.

What is the reciprocal of - 5 1/8? First, we convert to a negative improper fraction: - 5 1/8 = - 41/8, then we switch the numerator and denominator, and keep the same sign: - 8/41.

Visit the Math League

Introduction to Algebra

Variables
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Variables

A variable is a symbol that represents a number. Usually we use letters such as *n*, *t*, or *x* for variables. For example, we might say that *s* stands for the side-length of a square. We now treat *s* as if it were a number we could use. The perimeter of the square is given by $4 \times s$. The area of the square is given by $s \times s$. When working with variables, it can be helpful to use a letter that will remind you of what the variable stands for: let *n* be the number of people in a movie theater; let *t* be the time it takes to travel somewhere; let *d* be the distance from my house to the park.

Expressions

An expression is a mathematical statement that may use numbers, variables, or both.

Example:

The following are examples of expressions:

2

x

$3 + 7$

$2 \times y + 5$

$2 + 6 \times (4 - 2)$

$z + 3 \times (8 - z)$

Example:

Roland weighs 70 kilograms, and Mark weighs *k* kilograms. Write an expression for their combined weight. The combined weight in kilograms of these two people is the sum of their weights, which is $70 + k$.

Example:

A car travels down the freeway at 55 kilometers per hour. Write an expression for the distance the car will have traveled after *h* hours. Distance equals rate times time, so the distance traveled is equal to $55 \times h$..

Example:

There are 2000 liters of water in a swimming pool. Water is filling the pool at the rate of 100 liters per minute. Write an expression for the amount of water, in liters, in the swimming pool after *m* minutes. The amount of water added to the pool after *m* minutes will be 100 liters per minute times *m*, or $100 \times m$. Since we started with 2000 liters of water in the pool, we add this to the amount of water added to the pool to get the expression $100 \times m + 2000$.

To evaluate an expression at some number means we replace a variable in an expression with the number, and simplify the expression.

Example:

Evaluate the expression $4 \times z + 12$ when $z = 15$.

We replace each occurrence of *z* with the number 15, and simplify using the usual rules: parentheses first, then exponents, multiplication and division, then addition and subtraction.

$4 \times z + 12$ becomes

$4 \times 15 + 12 =$

$60 + 12 =$

Example:

Evaluate the expression $(1 + z) \times 2 + 12 \div 3 - z$ when $z = 4$.

We replace each occurrence of z with the number 4, and simplify using the usual rules: parentheses first, then exponents, multiplication and division, then addition and subtraction.

$(1 + z) \times 2 + 12 \div 3 - z$ becomes

$(1 + 4) \times 2 + 12 \div 3 - 4 =$

$5 \times 2 + 12 \div 3 - 4 =$

$10 + 4 - 4 =$

10.

Equations

An equation is a statement that two numbers or expressions are equal. Equations are useful for relating variables and numbers. Many word problems can easily be written down as equations with a little practice. Many simple rules exist for simplifying equations.

Example:

The following are examples of equations:

$2 = 2$

$17 = 2 + 15$

$x = 7$

$7 = x$

$t + 3 = 8$

$3 \times n + 12 = 100$

$w + 4 = 12 - w$

$y - 1 \div 2 - 9.3 = 34$

$3 \times (d + 4) - 11 = 321 - 2^3$

Example:

Translate the following word problem into an equation:

My age in years y plus 20 is equal to four times my age, minus 10.

The first expression stands for "my age in years plus 20", which is $y + 20$.

This is equal to the second expression for "four times my age, minus 10", which is $4 \times y - 10$.

Setting these two expressions equal to one another gives us the equation:

$y + 20 = 4 \times y - 10$

Solution of an Equation

When an equation has a variable, the solution to the equation is the number that makes the equation true when we replace the variable with its value.

Example:

We say $y = 3$ is a solution to the equation $4 \times y + 7 = 19$, for replacing each occurrence of y with 3 gives us

$4 \times 3 + 7 = 19$ ==>

$12 + 7 = 19$ ==>

$19 = 19$ which is true.

Examples:

$x = 100$ is a solution to the equation $x \div 2 - 40 = 10$

$z = 12$ is a solution to the equation $5 \times (z - 6) = 30$

Counterexample:

$y = 10$ is NOT a solution to the equation $4 \times y + 7 = 19$. When we replace each y with 10, we get

$4 \times 10 + 7 = 19$ ==>

$40 + 7 = 19$ ==>

$47 = 19$ not true!

Counterexamples:

$x = 200$ is NOT a solution to the equation $x \div 2 - 40 = 10$

$z = 20$ is NOT a solution to the equation $5 \times (z - 6) = 30$

Simplifying Equations

To find a solution for an equation, we can use the basic rules of simplifying equations. These are as follows:

1) You may evaluate any parentheses, exponents, multiplications, divisions, additions, and subtractions in the usual order of operations. When evaluating expressions, be careful to use the associative and distributive properties properly.

2) You may combine like terms. This means adding or subtracting variables of the same kind. The expression $2x + 4x$ simplifies to $6x$. The expression $13 - 7 + 3$ simplifies to 9.

3) You may add any value to both sides of the equation.

4) You may subtract any value from both sides of the equation. This is best done by adding a negative value to each side of the equation.

5) You may multiply both sides of the equation by any number except 0.

6) You may divide both sides of the equation by any number except 0.

Hint: Since subtracting any number is the same as adding its negative, it can be helpful to replace subtractions with additions of a negative number.

Example:

This problem illustrates grouping like terms and dealing with subtraction in an equation.

Solve $x - 12 + 20 = 37$.

Replacing the -12 with a +(-12), we get

$x + (-12) + 20 = 37.$

Since addition is associative, the two like terms (the integers) may be combined.

$(12) + 20 = 8$

The left side of the equation becomes

$x + 8 = 37.$

Now we may subtract 8 from each side of the equation, (we will actually add a -8 to each side).

$x + 8 + (-8) = 37 + (-8)$

$x + 0 = 29$

$x = 29$

We can check this solution in the original equation:

$29 - 12 + 20 = 37x + 0 = 29$

$17 + 20 = 37$

$37 = 37$ so our solution is correct.

Example:

This problem illustrates the proper use of the distributive property.

Solve $2 \times (x + 1 + 4) = 20.$

Grouping like terms in the parentheses, the left side of the equation becomes

$2 \times (x + 1 + 4)$ ==> $2 \times (x + 5).$

Using the distributive property,

$2 \times (x + 5)$ ==> $2 \times x + 2 \times 5.$

Carrying out multiplications,

$2 \times x + 2 \times 5$ ==> to $2x + 10.$

The equation now becomes

$2x + 10 = 20$.

Subtracting a 10 (adding a -10) to each side gives us

$2x + 10 + (-10) = 20 + (-10)$ ==>

$2x + (10 + (-10)) = 20 - 10$ ==>

$2x + 0 = 10$ ==>

$2x = 10$.

Since the x is multiplied by 2, we divide both sides by 2 to solve for x:

$2x = 10$ ==>

$2x \div 2 = 10 \div 2$ ==>

$(2x)/2 = 5$ ==>

$x = 5$.

We can check this solution in the original equation:

$2 \times (5 + 1 + 4) = 20$ ==>

$2 \times 10 = 20$ ==>

$20 = 20$ so our solution is correct.

Combining like terms

One of the most common ways to simplify an expression is to combine like terms. Numeric terms may be combined, and any terms with the same variable part may be combined.

Example:

Consider the expression $2 + 7x + 12 - 3x - 5$. The numeric like terms are the numbers 2, 12, and 5. The variable like terms are $7x$ and $3x$. Combining the numeric like terms, we have $2 + 12 - 5 = 14 - 5 = 9$. Combining the variable like terms, we have $7x - 3x = 4x$, so the expression $2 + 7x + 12 - 3x - 5$ simplifies to $9 + 4x$.

Simplifying with addition and subtraction

We can use addition and subtraction to get all the terms with variables on one side of an equation, and all the numeric terms on the other.

The equations $3x = 17$, $21 = y$, and $z/12 = 24$ each have a variable term on one side of the = sign, and a number on the other.

The equations $x + 3 = 12$, $21 = 30 - y$, and $(z + 2) \times 4 = 10$ do not.

We usually do this after simplifying each side using the distributive rules, eliminating parentheses, and combining like terms. Since addition is associative, it can be helpful to add a negative number to each side instead of subtracting to avoid mistakes.

Examples:

For the equation $3x + 4 = 12$, we can isolate the variable term on the left by subtracting a 4 from both sides:

$3x + 4 - 4 = 12 - 4 \implies$

$3x = 8$.

For the equation $7y - 200 = 10$, subtracting the 200 on the left side is the same as adding a -200:

$7y + (-200) = 10$.

If we add 200 to both sides of the equation, the 200 and -200 will cancel each other:

$7y + (-200) + 200 = 10 + 200 \implies$

$7y = 210$.

For the equation $8 = 20 - z$, we can add z to both sides to get $8 + z = 20 - z + z \implies 8 + z = 20$. Now subtracting 8 from both sides,

$8 + z - 8 = 20 - 8 \implies$

$z = 12$, so we get a solution for z.

Simplfying by multiplication

When solving for a variable, we want to get a solution like $x = 3$ or $z = 2001$. When a variable is divided by some number, we can use multiplication on both sides to solve for the variable.

Example:

Solve for x in the equation $x \div 12 = 5$.

Since the x on the left side is being divided by 12, the equation is the same as $x \times 1/12 = 5$. Multiplying both sides by 12 will cancel the 1/12 on the left side:

$x \times 1/12 \times 12 = 5 \times 12$ ==>

$x \times 1 = 60$ ==>

$x = 60$.

Simplifying by division

When solving for a variable, we want to get a solution like $x = 3$ or $z = 2001$. When a variable is multiplied by some number, we can use division on both sides to solve for the variable.

Example:

Solve for x in the equation $7x = 133$. Since the x on the left side is being multiplied by 7, we can divide both sides by 7 to solve for x:

$7x \div 7 = 133 \div 7$ ==>

$(7x)/7 = 133 \div 7$ ==>

$x/1 = 19$ ==>

$x = 19$.

Note that dividing by 7 is the same as multiplying both sides by 1/7.

Word problems as equations

When converting word problems to equations, certain "key" words tell you what kind of operations to use: addition, multiplication, subtraction, and division. The table below shows some common phrases and the operation to use.

Word	Operation	Example	As an equation
sum	addition	The sum of my age and 10 equals 27.	$y + 10 = 27$
difference	subtraction	The difference between my age and my younger sister's age, who is 11 years old, is 5 years.	$y - 11 = 5$
product	multiplication	The product of my age and 14 is 168.	$y \times 14 = 168$
times	multiplication	Three times my age is 60.	$3 \times y = 60$
less than	subtraction	Seven less than my age equals 32.	$y - 7 = 32$
total	addition	The total of my pocket change and 20 dollars is $22.43.	$y + 20 = 22.43$
more than	addition	Eleven more than my age equals 43.	$11 + y = 43$

Sequences

A sequence is a list of items. We can specify any item in the list by its place in the list: first, second, third, fourth, and so on. Many useful lists have patterns so we know what items occur in each place in the list. There are 2 kinds of sequences. A finite sequence is a list made up of a finite number of items. An infinite sequence is a list that continues without end.

Examples:

The following are examples of finite sequences.

The sequence 1, 3, 5, 7, 9, 11, 13, 15, 17, 19 is the sequence of the first 10 odd numbers.

The sequence *a, e, i, o, u,* is the sequence of vowels in the alphabet.

The sequence *m, m, m, m, m, m* is the sequence of 6 *m's*.

The sequence 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0 is the sequence of 12 alternating 1's and 0's.

The sequence 1, 2, 3, 4, ..., 9998, 9999, 10000 is the sequence of the first ten thousand integers.

The sequence 0, 1, 4, 9, 16, 25, 36, 49 is the sequence of the squares of the first 8 whole numbers.

Examples:

The following are examples of infinite sequences.

The sequence 2, 4, 6, 8, 10, 12, 14, 16, ... is the sequence of even whole numbers. The 100th place in this sequence is the number 200.

The sequence *a, b, c, a, b, c, a, b, c, a, b,* ... is the sequence of the letters a, b, c, repeating in this pattern forever.

The 100th place in this sequence is the letter *a*. The 300th place in this sequence is the letter *c*.

The sequence -1, 2, -3, 4, -5, 6, -7, 8, -9, ... is the sequence of integers with alternating signs. The 10th place in this sequence is 10. The 100th place in this sequence is 100. The 101st place in this sequence is -101.

The sequence 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, ... is a sequence of 1's separated by 1 zero, then 2 zeros, then 3 zeros, and so on. The 100th place in this sequence is a 0. The 105th place in this sequence is a 1.

The sequence 1, 3, 6, 10, 15, 21, 28, 36, 45, ... is the sequence of places the 1 occurs in the sequence of 1's and 0's above! If this sequence seems strange, note the difference between pairs of numbers next to one another:

3 - 1 = 2

6 - 3 = 3

10 - 6 = 4

15 - 10 = 5

21 - 15 = 6

28 - 21 = 7

Checking these differences makes the pattern clearer.

1, 1, 1, 1, 1, 1, ... is the sequence where every item in the list is the number 1.

1, 2, 3, 4, 5, 6, 7, ... is the sequence of counting numbers. Each item in the list is its place number in the list.

a, b, a, b, a, b, a, b, ... is the sequence of alternating letters a and b. The a's occur in odd-numbered places, and the b's occur in the even-numbered places.

1/1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, ... is the sequence of reciprocals of the whole numbers.

1, 4, 9, 16, 25, 36, 49, 64, 81, ... is the sequence of squares of the whole numbers.

a, e, i, o, u, a, e, i, o, u, a, e, ... is the repeating sequence of vowels in the alphabet.

4, 7, 10, 13, 16, 19, 22, 25, ... is the sequence of numbers beginning with the number 4, and each number in the list is 3 more than the number before it.
